Effective range theory of n-p scattering

J.S. Schwinger, H.A. Bethe and others developed a rigorous theory of the of the variation of scattering cross-section with energy in terms of two parameters i.e the scattering length (a) and another which also has the dimensions of length and is called the effective range (r_0). This theory, known as the effective range theory, which expresses the phase shift as a function of energy. We shall follow here closely the treatment of this theory by H.A. Bethe as it happens to be the simplest of all.

We can obtain by writing Schrodinger's equation for an S-wave and positive energy for the relative motion. By writing $u = r\psi$, the equation is

$$\frac{d^2u}{dr^2} + \left[K^2 - \frac{MV(r)}{\hbar^2}\right]u = 0 \; ; \qquad (1)$$

Where $K = (EM/\hbar^2)^{1/2}$. From this equation, ~~the~~ we can write the equations for two states of energy E_a and E_b respectively.

$$\frac{d^2u_a}{dr^2} + \left[K_a^2 - \frac{MV(r)}{\hbar^2}\right]u_a = 0 \; ; \qquad (2a)$$

$$\frac{d^2u_b}{dr^2} + \left[K_b^2 - \frac{MV(r)}{\hbar^2}\right]u_b = 0 \; ; \qquad (2b)$$

Multiplying equation (2a) by u_b and (2b) by u_a and then taking the difference, we get

$$u_a'' u_b - u_a u_b'' = (K_b^2 - K_a^2)\, u_a u_b \; ; \qquad (3)$$

where dashes represent differentiation w.r.t r.

By means of the identity,

$$u_a'' u_b - u_a u_b'' = \frac{d}{dr}\left[u_a' u_b - u_a u_b'\right] ; \qquad (4)$$

~~It may be written as:~~

Equation (3) may be written as.

$$\frac{d}{dr}(u_a' u_b - u_a u_b') = (K_b^2 - K_a^2)\, u_a u_b \; ; \qquad (5)$$

Integrating equation (5) between the limits 0 and ∞, we obtain,

$$\left[u_a' u_b - u_a u_b'\right]_{r=0}^{\infty} = (K_b^2 - K_a^2) \int_{r=0}^{\infty} u_a u_b \, dr \quad ; \text{———} (6)$$

The functions u_a and u_b satisfy the boundary condition

$$[u_a]_{r=0} = [u_b]_{r=0} = 0 \quad ; \text{———} (7)$$

Now for the sake of comparison, we introduce auxiliary wave-functions v_a and v_b which represent the asymptotic behaviour of u for large distance i.e which are the solutions to equations (2a,b) with the potential function $V(r)$ set equal to zero. Setting $V(r)=0$ amounts to considering the wavefunction outside the range of nuclear forces. Thus v_a and v_b are solutions to equations

$$v_a'' + K_a^2 v_a = 0 \quad ; \text{———} (8a)$$

and $$v_b'' + K_b^2 v_b = 0 \quad ; \text{———} (8b)$$

The wave functions v_a and v_b exist everywhere within and outside the nuclear force range but outside the nuclear force range, they behave exactly like u_a and u_b respectively and inside the nuclear force range they do not coincide with u_a and u_b.

~~Equations (8a,b) are similar to equation~~

The solutions of equations (8a,b) are of the form

$$v = B \sin(Kr + \delta_0)$$

While evaluating the normalization constant B, we require that for $r=0$, $v(0)=1$. This gives $B = 1/\sin\delta_0$ and hence,

$$v = \frac{\sin(Kr + \delta_0)}{\sin\delta_0} \quad ; \text{———} (9)$$

For large distances, u and v coincide and therefore

$$u(r) \underset{r\to\infty}{\longrightarrow} v(r) = \frac{\sin(Kr+\delta_0)}{\sin\delta_0} \quad ; \text{———} (10)$$

Now multiplying equation (8a) by v_b and (8b) by v_a, then subtracting and making use of the identity (equation 4), we get

$$\frac{d}{dr}(v_a' v_b - v_a v_b') = (K_b^2 - K_a^2) v_a v_b \quad ; \text{———} (11)$$

Integrating equation (11), we get

$$\left[v_a' v_b - v_a v_b'\right]_{r=0}^{\infty} = (K_b^2 - K_a^2) \int_{r=0}^{\infty} v_a v_b \, dr \quad ; \text{———} (12)$$

Subtracting (12) from equation (6), we get

(3)

$$\left[u_a' u_b - u_a u_b' - v_a' v_b + v_a v_b'\right]_{r=0}^{\infty} = (K_b^2 - K_a^2)\int_{r=0}^{\infty}(u_a u_b - v_a v_b)\,dr \quad (13)$$

Now since at distances beyond the range of nuclear forces, u and v become identical i.e $u_a = v_a$ and $u_b = v_b$ and therefore, there shall be no contribution to the L.H.S of equation (13) from the upper limit. At the lower limit i.e $r=0$, $u_a = u_b = 0$ (equation 4) and therefore this term does not appear on L.H.S. Consequently, equation (13.) reduces to

$$[v_a v_b' - v_a' v_b]_{r=0} = (K_b^2 - K_a^2)\int_0^{\infty}(v_a v_b - u_a u_b)\,dr \quad (14)$$

But since $[v_a]_{r=0} = [v_b]_{r=0} = 1$ as already assumed while finding out the value of the normalization constant B, hence, we have

$$[v_b' - v_a']_{r=0} = (K_b^2 - K_a^2)\int_0^{\infty}(v_a v_b - u_a u_b)\,dr \quad (15)$$

from equation (9), we have

$$v_a = \frac{\sin(K_a r + \delta_{oa})}{\sin\delta_{oa}} \quad \text{and} \quad v_b = \frac{\sin(K_b r + \delta_{ob})}{\sin\delta_{ob}}$$

$$v_a' = \frac{K_a\cos(K_a r + \delta_{oa})}{\sin\delta_{oa}} \quad \text{and} \quad v_b' = \frac{K_b\cos(K_b r + \delta_{ob})}{\sin\delta_{ob}}$$

$$\therefore [v_a']_{r=0} = K_a\cot\delta_{oa} \quad \text{and} \quad [v_b']_{r=0} = K_b\cot\delta_{ob}$$

substituting these values in equation (15) we get

$$K_b\cot\delta_{ob} - K_a\cot\delta_{oa} = (K_b^2 - K_a^2)\int_0^{\infty}(v_a v_b - u_a u_b)\,dr \quad (16)$$

considering the special case of $K_a \to 0$ i.e the case of zero energy, ~~then from equation~~

$$\lim_{K_a\to 0}(K_a\cot\delta_{oa}) = \frac{K_a}{\delta_{oa}} = -\frac{1}{a} \quad (17)$$

where 'a' is the scattering length for zero energy neutrons.

substing the value of (17) in equation (16) we get

$$K_b\cot\delta_{ob} + \frac{1}{a} = K_b^2\int_0^{\infty}(v_o v_b - u_o u_b)\,dr$$

dropping suffix b, we obtain

$$K\cot\delta_o + \frac{1}{a} = K^2\int_0^{\infty}(v_o v - u_o u)\,dr \quad (18)$$

This equation is exact and is the fundamental equation for the effective range theory.

In order to use it as a basis of approximation method, it is observed that the significant contributions to the integral on the R.H.S come from the region within the range of nuclear forces where u and u_0 differ appreciably from v and v_0. But just in that region their dependence on energy is very slight because the potential function $V(r)$ is much larger than E all through the low energy region upto about 10 MeV. Therefore it is a good approximation to replace u and v by their zero energy forms u_0 and v_0 throughout this region. Equation (18) then assumes the form

$$K\cot\delta_0 + \frac{1}{a} = K^2\int_0^\infty (v_0^2 - u_0^2)\,dr \quad ; \text{———} (19)$$

$$= \frac{K^2 r_0}{2} \quad ; \text{———} (20)$$

Where the parameter r_0 called the effective range.

$$r_0 = 2\int_0^\infty (v_0^2 - u_0^2)\,dr \quad ; \text{———} (21)$$

Equation (20) may also be written as

$$\cot\delta_0 = -\frac{1}{Ka} + \frac{1}{2}Kr_0 \quad ; \text{———} (22)$$

This expresses the phase shift δ_0 in the 'shape independent approximation. ~~From equations (22) and~~

from the equation of scattering cross-section

$$\sigma_{sc,0} = \frac{4\pi}{K^2}\sin^2\delta_0 \quad ; \text{———} (23)$$

Putting (22) in (23), we get

$$\sigma_{sc,0} = \frac{4\pi}{K^2 + K^2\cot^2\delta_0}$$

$$= \frac{4\pi}{K^2 + \left[-\frac{1}{a} + \frac{K^2 r_0}{2}\right]^2} \quad ; \text{———} (24)$$

This equation shows that the scattering cross-section, besides being a function of energy depends upon two parameters i.e the scattering length (a) and the effective range (r_0). Since the n-p scattering is spin dependent and hence four parameters come under the scattering cross-section i.e the singlet and triplet scattering lengths a_s and a_t and the singlet and triplet effective ranges i.e r_{0s} and r_{0t}.

~~From equations (24) and~~

$$\therefore \quad \sigma_0 = \frac{3\pi}{K^2 + \left(-\frac{1}{a_t} + \frac{K^2 r_{0t}}{2}\right)^2} + \frac{\pi}{K^2 + \left(-\frac{1}{a_s} + \frac{K^2 r_{0s}}{2}\right)^2} \qquad (25)$$

This is clearly independent of the form and shape of the nuclear potential. The four parameters used here are:

$a_t = 5.39F$, $r_{0t} = 1.70F$, $a_s = -23.7F$ and $r_{0s} = 2.7F$

Using these parameters in (25), we obtain $\sigma_{Th} = 20.41$ barn which is very close to the experimental value, $\sigma_{expt} = 20.36$ barn.

Where 1 Fermi $= 10^{-15}$ m

1 barn $= 10^{-28}$ m^2